Group-theoretic Description of Riemannian Spaces

Serhiy E. SAMOKHVALOV*

Department of Applied Mathematics

State Technical University, Dniprodzerzhinsk, Ukraine

April 22, 2007

Abstract

It is shown that a locally geometrical structure of arbitrarily curved Riemannian space is defined by a deformed group of its diffeomorphisms.

Key words: deformed group of diffeomorphisms, parallel transports, curvature, covariant derivatives, Riemannian space

Mathematics Subject Classification (2000): 53B05; 53B20; 58H05; 58H15

* **e**-*mail*: samokhval@dstu.dp.ua

Until recently it was thought impossible to realize Klein's Erlangen Program [1] for geometrical structures with arbitrary variable curvature; this Riemann-Klein antagonism, as it was figuratively called by E. Cartan [2], could only be overcome at the cost of program's modification and rejection of group structure of transformations which were used. Thus in [3] categories are employed while in [4] quasigroups are, and it is even stated that quasigroups are an algebraic equivalent of geometric notion of curvature.

In work [5] it was shown that group-theoretic description of connections in fiber bundles with arbitrary variable curvature can be performed by means of deformed infinite Lie groups introduced out of physical considerations in work [6], the structural equation follows from group axioms and is a necessary condition for existence of a group which defines given geometrical structure. This allowed realization of Klein's Program for connections in fiber bundles.

The structure of (pseudo)Riemannian space M is a special case of structure of affine connection in tangent bundle and therefore it can be specified similarly to arbitrary connection [5]. At the same time it necessary to apply additional conditions of torsion absence and coordination of connection with metric. The group fulfilling this description acts in tangent bundle of space M and is an infinite and specially deformed group which has the structure of semidirect product of diffeomorphisms group $\Gamma_T = Diff\ M$ and gauge group $SO\,(m, n-m)^g$, where n is space dimension [6].

It was shown in [6] that there exists a more natural way of group-theoretic description for (pseudo)Riemannian spaces, the one with the help of a narrower group, i.e. the deformed group Γ_T^H of diffeomorphisms of space M. The generators of such group define on M (locally, within the bounds of coordinate chart) field of affine vielbein, multiplication law define the rule of parallel transport of vectors, in consideration of which torsion is automatically zeroed in view of group axioms, components of vielbeins field in coordinate basis as well as anholonomity and connection coefficients are expressed through auxiliary deformation functions by means of which group Γ_T^H is built. Locally any space of torsion-free affine connection can be described in such fashion. With additional assumption that vielbeins field, defined by action of group Γ_T^H is (pseudo)orthonormal, and in case of parallel transport of vectors they merely rotate, coefficients of affine connection in coordinate basis automatically become Christoffel symbols, i.e. they are defined through metric in a certain way, therefore there is no need to postulate this statement.

Publication [6] had physical value and group-theoretical and geometrical aspects were only slightly touched upon there, while some important geometrical relations were neglected at all. This work makes up for this. Specifically, we show that definition of curvature tensor (which in our approach becomes a characteristic of group Γ_T^H) through connection coefficients follows from equation which comes from group axioms and is an essential condition for existence of group Γ_T^H.

With the help of groups Γ_T^H Klein's Erlangen Program is realized for (pseudo)Riemannian

spaces of arbitrary variable curvature, in the most rational fashion at that. Groups Γ_T^H act on M and their transformations are interpreted as gauge translation in curved (pseudo)Riemannian spaces. It is due to this that group-theoretic description of (pseudo)Riemannian spaces through groups Γ_T^H is important for gravitation theory, gravitation being interpreted as gauge theory of translations group [7].

The work doesn't deal with global topological problems and all relations are obtained within the bounds of a single coordinate chart. Besides, we perceive groups to be respective local groups.

1. Let's specify the general procedure of building deformed infinite Lie groups [5] for the case of deformed group of diffeomorphisms Γ_T^H. This time, opposite to [5] we will use coordinate approach.

Let O be a coordinate chart on manifold M with coordinates x^μ (we use Greek alphabet for indices). We assume coordinates to be fixed and won't change them further.

In O there acts Abelian group of translations $T = \{\tilde{t}\}$ according to the formula:

$$x'^\mu = x^\mu + \tilde{t}^\mu$$

In set $C_\infty(O, T)$ of smooth mappings of O in T let's single out subset $\Gamma_T = \{\tilde{t}(x)\}$ with condition:

$$det\{\delta^\mu_\nu + \partial_\nu \tilde{t}^\mu(x)\} \neq 0, \ \forall x \in O,$$

where $\partial_\nu := \partial/\partial x^\nu$, and assign to it the multiplication law $\tilde{t''} = \tilde{t} \times \tilde{t'}$:

$$\tilde{t''}^\mu(x) = \tilde{t}^\mu(x) + \tilde{t'}^\mu(x'), \tag{1}$$

where

$$x'^\mu = x^\mu + \tilde{t}^\mu(x). \tag{2}$$

With it the set Γ_T becomes a local group. Group Γ_T acts smoothly in chart O according to the formula (2) and is a local group of diffeomorphisms of chart O in additive parameterization. According to definition 1 from [5] group Γ_T is *a group of undeformed chart O, or undeformed group.*

Let's deform group Γ_T by means of *deformation* H which is defined by mapping $H : O \times T \to T$ with properties which in our case are described as follows:

$1H)$ $H \in C_\infty(O, T)$;

$2H)$ $H(x, 0) = 0, \ \forall x \in O$;

$3H)$ $\exists$ mapping $K : O \times T \to T : K(x, H(x, \tilde{t})) = \tilde{t}, \ \forall x \in O, \ \tilde{t} \in T.$

Group $\Gamma_T^H = \{t(x)\}$ is obtained from group Γ_T by isomorphism, which is specified by deformation H according to the formula:

$$t^m(x) = H^m(x, \tilde{t}(x)). \tag{3}$$

Functions $t^m(x)$ which parameterize group Γ_T^H (we use indices from Latin alphabet for them), satisfy the condition:

$$det\{\delta_\nu^\mu + d_\nu K^\mu(x, t(x))\} \neq 0, \ \forall x \in O,$$

where $d_\nu := d/dx^\nu$, and multiplication law $t'' = t * t'$ is defined by isomorphism (3):

$$t''^m(x) = \varphi^m(x, t(x), t'(x')) := H^m(x, K(x, t(x)) + K(x', t'(x'))), \tag{4}$$

where

$$x'^\mu = f^\mu(x, t(x)) := x^\mu + K^\mu(x, t(x)). \tag{5}$$

Group Γ_T^H acts smoothly in the chart O according to the formula (5). According to definition 3 from [5] group Γ_T^H is *a group of deformed chart O*, or *deformed group*.

Multiplication law (4) for deformed group Γ_T^H explicitly depends on x, and, therefore, structural constants analogue for groups Γ_T^H is structure functions $F(x)^n_{kl}$, which are defined by the formula:

$$F(x)^n_{kl} := \left(\partial^2_{k,l'} - \partial^2_{l,k'}\right) \varphi^n(x, t, t')\Big|_{t=t'=0}. \tag{6}$$

(here and henceforth $\partial_k := \partial/\partial t^k$, primed index stands for differentiation with respect to t').

2. Let's introduce auxiliary functions:

$$h(x)^m_\mu = \frac{\partial}{\partial \tilde{t}^\mu} H^m(x, \tilde{t})\Big|_{\tilde{t}=0}.$$

Property $3H$ allows fulfillment of condition:

$$det\left\{h(x)^m_\mu\right\} \neq 0, \quad \forall x \in O, \tag{7}$$

wherefrom there follows existence of functions $h(x)^\mu_m$ of the type that $h(x)^\mu_n h(x)^m_\mu = \delta^m_n$, $\forall x \in O$. It is obvious that $h(x)^\mu_m = \partial_m K^\mu(x, t)\Big|_{t=0}$. With the help of these functions we will substitute Greek indices for Latin and vice versa.

Assuming parameters in multiplication law for group Γ_T^H to be constant, let's define functions:

$$\mu(x, t)^m{}_n := \partial_{n'} \varphi^m(x, t', t)\Big|_{t'=0}, \tag{8}$$

$$\lambda(x, t)^m{}_n := \partial_{n'} \varphi^m(x, t, t')\Big|_{t'=0}. \tag{9}$$

The condition of multiplication law associativity in group Γ_T^H: $(t*t')*t'' = t*(t'*t'')$ is fulfilled automatically for any deformation H in view of multiplication law (1) associativity in diffeomorphisms group. Let's derive this condition for constant parameters of group Γ_T^H:

$$\varphi^m(x,\varphi(x,t,t'),t'') = \varphi^m(x,t,\varphi(x',t',t'')). \tag{10}$$

Differentiating it with respect to t in zero we obtain the equation:

$$h(x)_k^\mu \partial_\mu \varphi^m(x,t,t') - \mu(x,t)^n k \partial_n \varphi^m(x,t,t') = -\mu(x,\varphi(x,t,t'))^m{}_k, \tag{11}$$

and with respect to t'' in zero the equation:

$$\lambda(x',t')^n{}_k \partial_{n'} \varphi^m(x,t,t') = \lambda(x,\varphi(x,t,t'))^m{}_k. \tag{12}$$

The condition for their integrabilily is equation:

$$h(x)_k^\nu \partial_\nu \mu(x,t)^m{}_l - \mu(x,t)^n{}_k \partial_n \mu(x,t)^m{}_l - h(x)_l^\nu \partial_\nu \mu(x,t)^m{}_k + \mu(x,t)^n{}_l \partial_n \mu(x,t)^m{}_k =$$
$$= F(x)_{kl}^n \mu(x,t)^m{}_n \tag{13}$$

and

$$\lambda(x,t)^n{}_k \partial_n \lambda(x,t)^m{}_l - \lambda(x,t)^n{}_l \partial_n \lambda(x,t)^m{}_k = F(x')_{kl}^n \lambda(x,t)^m{}_n. \tag{14}$$

respectively.

Let's call equations (11) and (12) *the left and the right Lie equation for groups* Γ_T^H, while equations (13) and (14) *the left and the right Maurer-Cartan equations for groups* Γ_T^H.

If the condition of associativity (10) is immediately differentiated with respect to t' and t'' in zero with differing sequence we obtain the equation:

$$h(x)_k^\nu \partial_\nu \lambda(x,t)^m{}_l - \mu(x,t)^n{}_k \partial_n \lambda(x,t)^m{}_l + \lambda(x,t)^n{}_l \partial_n \mu(x,t)^m{}_k = 0. \tag{15}$$

Let's perform consequently two Γ_T^H-transformations with constant parameters t and t'. Composition law of transformations results in equation:

$$f^\mu(f(x,t),t') = f^\mu(x,\varphi(x,t,t')),$$

which is fulfilled automatically for any deformation H in view of performance of composition law in the group of diffeomorphisms Γ_T. Differentiating it with respect to t in zero we obtain the equation:

$$h(x)_k^\nu \partial_\nu f^\mu(x,t) - \mu(x,t)^n{}_k \partial_n f^\mu(x,t) = 0, \tag{16}$$

and differentiating it with respect to t' in zero the equation:

$$h(x')_k^\mu - \lambda(x,t)^n{}_k \partial_n f^\mu(x,t) = 0. \tag{17}$$

The condition of integrabilily of these equations in case of fulfillment of equations (13) and (14) is equation:

$$h(x)^{\nu}_{k}\partial_{\nu}h(x)^{\mu}_{l} - h(x)^{\nu}_{l}\partial_{\nu}h(x)^{\mu}_{k} = F(x)^{n}_{kl}h(x)^{\mu}_{n}. \tag{18}$$

We will call equations (16) and (17) *the left and the right Lie equations for groups Γ^H_T transformations*, while equation (18) *the Maurer-Cartan equation for groups Γ^H_T transformations.*

3. Let's introduce differentiating operators:

$$X^{\tau}_{k} = h(x)^{\nu}_{k}\partial_{\nu} - \mu(x,t)^{n}{}_{k}\partial_{n},$$

$$X^{\upsilon}_{k} = \lambda(x,t)^{n}{}_{k}\partial_{n},$$

which we will call *generators of leftward and rightward shifts, or horizontal and vertical generators* of group Γ^H_T respectively, as well as

$$X_k = h(x)^{\nu}_{k}\partial_{\nu}$$

- *generators of action of group Γ^H_T on O.* In terms of generators, equations (13) - (15) as well as equation (18) have quite an elegant form:

$$[X^{\tau}_{k}, X^{\tau}_{l}] = F(x)^{n}_{kl}X^{\tau}_{n}, \tag{19}$$

$$[X^{\upsilon}_{k}, X^{\upsilon}_{l}] = F(x')^{n}_{kl}X^{\upsilon}_{n}, \tag{20}$$

$$[X^{\tau}_{k}, X^{\upsilon}_{l}] = 0, \tag{21}$$

$$[X_k, X_l] = F(x)^{n}_{kl}X_n, \tag{22}$$

where square brackets stand for operators commutator. These equations follow from multiplication law associativity for group Γ^H_T, however, due to its infinity, generators commutators are expanded into generators not by means of structure constants as in finite parametric Lie groups, but by means of structure functions dependent on x.

The condition for integrability of equations (19) - (22) is the equation for structure functions of group Γ^H_T:

$$h(x)^{\nu}_{k}\partial_{\nu}F(x)^{n}_{lm} + F(x)^{n}_{kp}F(x)^{p}_{lm} + \mathrm{cycle}(klm) = 0, \tag{23}$$

which is derived from Jacobi's identity for dual generators commutator.

4. Let's study the expansion of functions defined by formulae (8), (9) according to group parameters with accuracy to the second order inclusive:

$$\mu(x,t)^{m}{}_{n} = \delta^{m}_{n} + \gamma^{m}{}_{nk}t^{k} + \frac{1}{2}\rho^{m}{}_{lkn}t^{l}t^{k}, \tag{24}$$

$$\lambda(x,t)^{m}{}_{n} = \delta^{m}_{n} + \gamma^{m}{}_{kn}t^{k} + \frac{1}{2}\sigma^{m}{}_{lkn}t^{l}t^{k}. \tag{25}$$

The coefficients of these expansions ${\gamma^m}_{nk}$, ${\rho^m}_{nkl}$ and ${\sigma^m}_{nkl}$ depend on x in general case; however, this dependence, defined by deformation functions, will be specified a little later and to make it shorter we will not show explicitly neither to the coefficients themselves nor to the functions they define. On inserting expansions (24) and (25) into formulae (13) and (14) we arrive at the result that in zeroth order with respect to t the structure functions of group Γ_T^H are defined by skew-symmetric part of coefficients ${\gamma^m}_{kn}$:

$$F^m_{kn} = {\gamma^m}_{kn} - {\gamma^m}_{nk}. \tag{26}$$

This formula follows directly from definition (6) for structure functions of group Γ_T^H and actually can be considered their definition. Let's introduce functions:

$${R^m}_{lkn} := {\rho^m}_{lkn} - {\rho^m}_{lnk}, \tag{27}$$

$${S^m}_{lkn} := {\sigma^m}_{lkn} - {\sigma^m}_{lnk}, \tag{28}$$

which we will call *tensors of left and right curvature of group* Γ_T^H respectively. In the first order with respect to t from formula (13) we derive:

$${R^m}_{lkn} = -{\gamma^m}_{sl}F^s_{kn} + h^\sigma_k\partial_\sigma{\gamma^m}_{nl} - h^\sigma_n\partial_\sigma{\gamma^m}_{kl} + {\gamma^m}_{ks}{\gamma^s}_{nl} - {\gamma^m}_{ns}{\gamma^s}_{kl}, \tag{29}$$

and from formula (14)

$${S^m}_{lkn} = {\gamma^m}_{ls}F^s_{kn} + h^\sigma_l\partial_\sigma F^m_{kn} + {\gamma^m}_{sk}{\gamma^s}_{ln} - {\gamma^m}_{sn}{\gamma^s}_{lk}. \tag{30}$$

Relation (15) yields:

$${\sigma^m}_{lkn} - {\rho^m}_{lnk} = h^\sigma_k\partial_\sigma{\gamma^m}_{ln} + {\gamma^m}_{ks}{\gamma^s}_{ln} - {\gamma^m}_{sn}{\gamma^s}_{kl},$$

wherefrom follows:

$${R^m}_{lkn} + {S^m}_{lkn} = h^\sigma_k\partial_\sigma{\gamma^m}_{ln} - h^\sigma_n\partial_\sigma{\gamma^m}_{lk} + {\gamma^m}_{ks}{\gamma^s}_{ln} - {\gamma^m}_{sn}{\gamma^s}_{kl} + {\gamma^m}_{sk}{\gamma^s}_{nl} - {\gamma^m}_{ns}{\gamma^s}_{lk}. \tag{31}$$

Taking into account formulae (26), (29) and (30), expression (31) is the result of condition (23).

5. The relations derived so far, follow solely from group axioms without consideration of deformation mode of building the group Γ_T^H which allows their fulfillment. However, both multiplication law (4) and action (5) of deformed group Γ_T^H in chart O are defined by deformation H with the help of which it is built. Let's express auxiliary functions of group Γ_T^H through deformation functions. To this end, let's introduce matrices $H(x,t)^m_\mu = \dfrac{\partial}{\partial\tilde{t}^\mu}H^m(x,\tilde{t})\Big|_{\tilde{t}=K(x,t)}$. Matrices $H(x,t)^\mu_m = \partial_m K^\mu(x,t)$ will be inverse to them. Direct use of the second equality in (4) in definitions (8) and (9) yields:

$$\mu(x,t)^m{}_n = H(x,t)^m_\mu(\delta^\mu_\nu + \partial_\nu K^\mu(x,t))h(x)^\nu_n, \tag{32}$$

$$\lambda(x,t)^m{}_n = H(x,t)^m_\mu h(x+K(x,t))^\mu_n, \tag{33}$$

or depending upon $\tilde{t}$:

$$\mu(x,\tilde{t})^m{}_n = \frac{\partial}{\partial \tilde{t}^\mu} H^m(x,\tilde{t})(\delta^\mu_\nu + \partial_\nu \tilde{t}^\mu) h(x)^\nu_n,$$

$$\lambda(x,\tilde{t})^m{}_n = \frac{\partial}{\partial \tilde{t}^\mu} H^m(x,\tilde{t}) h(x+\tilde{t})^\mu_n. \tag{34}$$

Let's consider the expansion of functions of deformation H up to the third order with respect to $\tilde{t}$ inclusive:

$$H^m(x,\tilde{t}) = h^m_\mu(\tilde{t}^\mu + \frac{1}{2}\Gamma^\mu_{\nu\rho}\tilde{t}^\nu\tilde{t}^\rho + \frac{1}{6}\Delta^\mu_{\nu\rho\sigma}\tilde{t}^\nu\tilde{t}^\rho\tilde{t}^\sigma). \tag{35}$$

Coefficients h^m_μ satisfy condition (7) and $\Gamma^\mu_{\nu\rho}$, $\Delta^\nu_{\nu\rho\sigma}$ are symmetric in lower indices. On fulfilling these conditions, the coefficients of expansion (35) are arbitrary smooth functions of x. Applying them, with accuracy to the second order with respect to t we derive:

$$K^\mu(x,t) = h^\mu_k t^k - \frac{1}{2}\Gamma^\mu_{kl} t^k t^l,$$

$$H(x,t)^m_\mu = h^m_\mu + \Gamma^m_{\mu k} t^k + \frac{1}{2}(\Delta^m_{\mu kl} - \Gamma^m_{\mu s}\Gamma^s_{kl}) t^k t^l.$$

In consideration of these expansions formulae (32) and (33) give the following expressions for coefficients of expansions (24) and (25) through coefficients of expansion (35):

$$\gamma^m{}_{kn} = h^m_\mu(\Gamma^\mu_{kn} + h^\nu_k \partial_\nu h^\mu_n), \tag{36}$$

$$\rho^m{}_{lkn} = h^m_\mu(\Delta^\mu_{lkn} - \Gamma^\mu_{ns}\Gamma^s_{kl} - h^\nu_n \partial_\nu \Gamma^\mu_{\kappa\lambda} h^\kappa_k h^\lambda_l), \tag{37}$$

$$\sigma^m{}_{lkn} = h^m_\mu(\Delta^\mu_{lkn} - \Gamma^\mu_{ns}\Gamma^s_{kl} + h^\kappa_k h^\lambda_l \partial^2_{\kappa\lambda} h^\mu_n - \Gamma^\nu_{kl}\partial_\nu h^\mu_n + (\Gamma^\mu_{k\sigma} h^\nu_l + \Gamma^\mu_{l\sigma} h^\nu_k)\partial_\nu h^\sigma_n).$$

Inserting these expressions into definitions (26) - (28) and taking into account the symmetry in lower indices of coefficients Γ^μ_{kn} and Δ^μ_{lkn} we derive formula (18) for structure functions of group Γ^H_T, and for its curvature tensors the formulae as follows:

$$R^m{}_{lkn} = h^m_\mu(\partial_\kappa \Gamma^\mu_{\nu\lambda} - \partial_\nu \Gamma^\mu_{\kappa\lambda} + \Gamma^\mu_{\kappa\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\mu_{\nu\sigma}\Gamma^\sigma_{\kappa\lambda}) h^\lambda_l h^\kappa_k h^\nu_n, \tag{38}$$

$$S^m{}_{lkn} = h^m_\mu(\Gamma^\mu_{l\sigma} F^\sigma_{kn} + h^\sigma_l \partial_\sigma F^\mu_{kn} + \Gamma^\mu_{k\sigma}\Gamma^\sigma_{nl} - \Gamma^\mu_{n\sigma}\Gamma^\sigma_{kl} + \Gamma^\sigma_{nl}\partial_\sigma h^\mu_k - \Gamma^\sigma_{kl}\partial_\sigma h^\mu_n +$$
$$h^\sigma_l(\Gamma^\mu_{k\sigma}\partial_\lambda h^\sigma_n - \Gamma^\mu_{n\sigma}\partial_\lambda h^\sigma_k + \partial_\lambda h^\sigma_n \partial_\sigma h^\mu_k - \partial_\lambda h^\sigma_k \partial_\sigma h^\mu_n)),$$

These formulae could be derived directly from formulae (29), (30) on inserting expressions (36) and (18) into them. The reason for this is that the condition of multiplication law associativity in groups Γ^H_T, which yields equations (13) and (14) wherefrom formulae (29) and (30) were derived, is fulfilled automatically through deformation mode of building groups Γ^H_T which we apply.

6. The very form of tensor of left curvature of group Γ_T^H (formulae (29) or (38)) as well as that of its other characteristics indicates that groups Γ_T^H possess ample geometric data which we proceed to study below.

Generators $X_m = h_m^\mu \partial_\mu$ of action of group Γ_T^H specify on O a field of affine vielbeins, auxiliary functions of deformation h_m^μ transfer from coordinate to affine bases. Elements t of group Γ_T^H specify on O *vector fields* $t = t^m X_m$, parameters t^m of group Γ_T^H are components of these fields in basis X_m.

Structure functions of group Γ_T^H with lower coordinate indices in view of formula (18) can be represented as:

$$F^k_{\mu\nu} = \partial_\nu h^k_\mu - \partial_\mu h^k_\nu,$$

thus they have geometric meaning (with accuracy to factor -2) of *anholonomity object*.

Let's study multiplication law $t*\tau$ in group Γ_T^H for the case of infinitesimal second multiplier:

$$(t*\tau)^m(x) = t^m(x) + \lambda(x, t(x))^m{}_n \tau^n(x'),$$

where $x'^\mu = f^\mu(x, t(x))$. Thus, this law gives the rule for composition of vectors fitted in different points, or the *rule of parallel transport* of vector field τ from point x' to point x:

$$\tau_\parallel^m(x) = \lambda(x, t(x))^m{}_n \tau^n(x'). \tag{39}$$

Taking t to be infinitesimal as well, and taking into account expansion (25) we have:

$$\tau_\parallel^m(x) = \tau^m(x) + t^n(x)\nabla_n \tau^m(x), \tag{40}$$

where

$$\nabla_n \tau^m(x) = h_n^\sigma \partial_\sigma \tau^m(x) + \gamma^m{}_{nk}\tau^k(x)$$

by definition is a *covariant derivative* of vector field τ to the direction X_n. Thus, functions $\gamma^m{}_{nk}$ which define the second of parameters order of multiplication law in group Γ_T^H get geometric meaning of *coefficients of affine connection in basis* X_n.

In coordinate basis, relation (39) in view of (34) becomes:

$$\tau_\parallel^\mu(x) = \frac{\partial}{\partial \tilde{t}^\nu} H^\mu(x, \tilde{t})\tau^\nu(x + \tilde{t}),$$

or in case of infinitesimal $\tilde{t}$:

$$\tau_\parallel^\mu(x) = \tau^\mu(x) + \tilde{t}^\nu(x)\nabla_\nu \tau^\mu(x),$$

in relation to which

$$\nabla_\nu \tau^\mu(x) = \partial_\nu \tau^\mu(x) + \Gamma^\mu_{\sigma\nu}\tau^\sigma(x).$$

Thus, coefficients $\Gamma^\mu_{\sigma\nu}$ which define the second order of expansion (35) of deformation functions get geometric meaning of *coefficients of affine connection in coordinate basis*. They are arbitrary smooth functions symmetric in lower indices, corresponding to arbitrary torsion-free affine connection. For undeformed group Γ^H_T covariant derivatives is obviously congruent with partial derivatives.

It is in this meaning that, specifying the rule of parallel transport of vectors by its multiplication law (which is defined by deformation H), deformed groups of diffeomorphisms Γ^H_T specify by their action a structure of torsion-free affine connection in tangent bundle of chart O; arbitrary torsion-free affine connection can be specified over O in such fashion.

The same connection is specified by all groups $\Gamma^{H'}_T$ in which coefficients $\Gamma^\mu_{\sigma\nu}$ in the second order of expansion (35) of their function of deformation H' are congruent, particularly if

$$H'^m(x,\tilde{t}) = L(x)^m{}_n H^n(x,\tilde{t}). \tag{41}$$

where matrices $L(x)^m{}_n$, dependent upon x, belong to gauge group $GL(n)^g$. In transformation (41) the affine vielbeins field on O changes: $X'_m = L^{-1}(x)^n{}_m X_n$. The third and higher orders of parameter expansion of deformation functions do not influence the connection and can be arbitrary. This is related to the fact that definition (39) allows to make parallel transport of vector field from point x' to point x for *finite* distance $\tilde{t}(x) = x' - x = K(x, t(x))$, though infinitesimal shifts are enough to specify a connection. There are, however, quite natural additional requirements to deformation functions which follow from geometric point of view and allow to completely fix deformation functions with respect to the first two orders of expansion (35), i.e. with respect to the affine vielbeins field and affine connection coefficients. They are related to the generation of finite parallel transports (39) with the help of integral sequence of infinitesimal transports (40), and we make plans to study this problem for the structure of affine connection in our next work [8].

Let's choose points $x_1 = x + \tilde{t}_1$, $x_2 = x + \tilde{t}_2$, $x_3 = x + \tilde{t}_1 + \tilde{t}_2$ ($\tilde{t}_1$ and $\tilde{t}_2$ we assume to be constant) and perform, according to formula (39), parallel transport of vector field $\tau(x)$ from point x_3 to point x_1, and then to point x (first choice), as well as from point x_3 to point x_2 and then to point x (second choice). The difference of the results obtained gives:

$$\tau^m_\parallel(x)_1 - \tau^m_\parallel(x)_2 = (\lambda(x,\tilde{t}_1)^m{}_k \lambda(x_1,\tilde{t}_2)^k{}_n - \lambda(x,\tilde{t}_2)^m{}_k \lambda(x_2,\tilde{t}_1)^k{}_n)\tau^n(x_3).$$

For infinitesimal $\tilde{t}_1$ and $\tilde{t}_2$ using formulae (34), (35) and (38) we derive:

$$\tau^m_\parallel(x)_1 - \tau^m_\parallel(x)_2 = R^m{}_{n\rho\sigma}\tau^n(x)\tilde{t}^\rho_1 \tilde{t}^\sigma_2.$$

Thus, the tensor $R^m{}_{n\rho\sigma}$ of the left curvature of group Γ^H_T, which according to the formula (27) is a skew-symmetric part of coefficients $\rho^m{}_{n\rho\sigma}$, which (partially) define the third of parameter order of multiplication law in group Γ^H_T, acquires the geometric meaning of *curvature tensor* of affine connection structure, which is specified in O by the action of group Γ^H_T.

Let's summarize the obtained results.

Theorem 1. *Deformed group Γ_T^H of diffeomorphisms in chart O specifies by its action on O an affine vielbeins field and structure of torsion-free affine connection in tangent bundle over O. Geometric characteristics of space O, such as anholonomity object, affine connection coefficients, curvature tensor are defined by multiplication law in group Γ_T^H, which, in its turn, is defined by deformation H, with the help of which group Γ_T^H is built.*

Arbitrary torsion-free affine connection can be specified over O in such fashion.

Thus, geometric structure of torsion-free affine connection with arbitrary variable curvature can be referred to only in terms of deformed groups Γ_T^H of diffeomorphisms, due to which Klein's Erlangen Program is realized for such structure, the condition of torsion absence (26) is fulfilled in view of group axioms and there is no need for its additional application.

7. Let's now assume that matrices $\lambda(x,t)^m{}_n$ belong to the gauge group $SO(m,n-m)^g$, so they satisfy the equation:

$$\lambda(x,t)^k{}_m \lambda(x,t)^l{}_n \eta_{kl} = \eta_{mn}, \tag{42}$$

where η_{mn} is a flat metric (with the help of which we will lowering indices). This means that vielbein field X_m, specified by the action of group Γ_T^H is (pseudo)orthonormal and in case of parallel transport of vectors (39) they merely (pseudo)rotate. Thus, the action of group Γ_T^H in O specifies the structure of (pseudo)Riemann space with metric $g_{\mu\nu} = h_\mu^m h_\nu^n \eta_{mn}$.

In the first order with respect to t the equation (42) produces:

$$\gamma^{\cdot}_{ksl} + \gamma^{\cdot}_{lsk} = 0,$$

which allows, with the use of definition (26), to express affine connection coefficients in vielbein basis in terms of structure functions of group Γ_T^H:

$$\gamma^{\cdot}_{slk} = \frac{1}{2}\left(F^{\cdot}_{slk} + F^{\cdot}_{ksl} + F^{\cdot}_{lsk}\right). \tag{43}$$

Recalling geometric interpretation of structure functions we can see that coefficients $\gamma^s{}_{lk}$ in this case become *Ricci rotation coefficients.*

With the use of formula (34) equation (42) becomes equation directly for deformation functions:

$$\frac{\partial}{\partial\tilde{t}^\mu} H^m(x,\tilde{t}) \frac{\partial}{\partial\tilde{t}^\nu} H^n(x,\tilde{t}) \eta_{mn} = g(x+\tilde{t})_{\mu\nu}. \tag{44}$$

Besides, we have relation $2H$ for the function H:

$$H^m(x,0) = 0, \tag{45}$$

which we will consider a boundary condition for differential equation (44). The solution to the problem (44), (45) allows to find deformation functions $H^m(x,\tilde{t})$ with whose help group

Γ_T^H is produced; the group specifies in O a structure of (pseudo)Riemannian space, arbitrary (pseudo)Riemannian structure can be specified in O in such fashion.

Equation (44) is invariant under given transformations:

$$H'^m(x,\tilde{t}) = \Lambda(x)^m{}_n H^n(x,\tilde{t}), \tag{46}$$

where matrices $\Lambda(x)^m{}_n$, dependent upon x, belong to gauge group $SO(m,n-m)^g$, i.e. satisfy relation $\Lambda(x)^k{}_m \Lambda(x)^l{}_n \eta_{kl} = \eta_{mn}$. Thus, if the equation (44) is satisfied by deformation functions $H^m(x,\tilde{t})$, it is also satisfied by functions $H'^m(x,\tilde{t})$, which are defined by formula (46) with arbitrary $\Lambda(x)^m{}_n$ from the group $SO(m,n-m)^g$. All such groups $\Gamma_T^{H'}$ specify the same (pseudo)Riemann structure on O.

From geometric point of view, the field of (pseudo)orthonormal vielbeins: $X'_m = \Lambda^{-1}(x)^n{}_m X_n$ changes during transformations (46). By the field of (pseudo)orthonormal vielbeins X_m from equation (44) deformation functions are *uniquely* defined (let's recall that we assume coordinates in O to be fixed).

Let's point out that in our approach in (pseudo)Riemannian space with respect to the field of (pseudo)orthonormal vielbeins X_m the rule of parallel transport of vectors to finite distance $\tilde{t}(x) = x' - x = K(x,t(x))$ is uniquely defined. On the other hand, in general case of curved space the result of parallel transport depends upon the curve along which it is performed. So there is a question to be asked: along which curve connecting points x' and x in the general case of curved (pseudo)Riemannian space during performance of integral sequence of infinitesimal transports (40) do we get the result which is given by formula (39)? We plan to study this problem in our next publication [8].

In the first order with respect to $\tilde{t}$ equation (44) produces:

$$\Gamma^{\boldsymbol{\cdot}}_{\mu\nu\sigma} + \Gamma^{\boldsymbol{\cdot}}_{\nu\mu\sigma} = \partial_\sigma g_{\mu\nu}, \tag{47}$$

which, in view of symmetry of coefficients $\Gamma^\sigma_{\mu\nu}$ in lower indices, gives formula:

$$\Gamma^{\boldsymbol{\cdot}}_{\sigma\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}), \tag{48}$$

which, naturally, could be derived as the result of formula (43) in consideration of relation (36). Formula (48) indicates that $\Gamma^{\boldsymbol{\cdot}}_{\sigma\mu\nu}$ and $\Gamma^\sigma_{\mu\nu}$ in our case become *Christoffel symbols of the I^{st} and II^{nd} type* respectively.

In the second order with respect to $\tilde{t}$ it follows from equation (44) that:

$$\Delta^{\boldsymbol{\cdot}}_{\mu\nu\sigma\rho} + \Delta^{\boldsymbol{\cdot}}_{\nu\mu\sigma\rho} = \partial^2_{\sigma\rho} g_{\mu\nu} - \Gamma^{\boldsymbol{\cdot}}_{\tau\mu\sigma}\Gamma^\tau_{\nu\rho} - \Gamma^{\boldsymbol{\cdot}}_{\tau\nu\sigma}\Gamma^\tau_{\mu\rho},$$

which, in view of symmetry of coefficients $\Delta^\sigma_{\mu\nu\rho}$ in lower indices and relation (47), gives:

$$\Delta^\sigma_{\mu\nu\rho} = \frac{1}{3}(\partial_\rho \Gamma^\sigma_{\mu\nu} + \partial_\nu \Gamma^\sigma_{\mu\rho} + \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\sigma_{\tau\rho}\Gamma^\tau_{\mu\nu} + \Gamma^\sigma_{\tau\nu}\Gamma^\tau_{\mu\rho} + \Gamma^\sigma_{\tau\mu}\Gamma^\tau_{\nu\rho}).$$

Inserting this expression into formula (37) and considering formula (38) we derive the expression

$$\rho^{\sigma}{}_{\mu\nu\rho} = \frac{1}{3}(R^{\sigma}{}_{\mu\nu\rho} + R^{\sigma}{}_{\nu\mu\rho}),$$

the insertion of which into definition (27) produces a well-known identity for curvature tensor:

$$R^{\sigma}{}_{\mu\nu\rho} + R^{\sigma}{}_{\nu\rho\mu} + R^{\sigma}{}_{\rho\mu\nu} = 0.$$

Thus we have proved

Theorem 2. *Deformed group Γ_T^H of diffeomorphisms of chart O, produced with the help of deformation, functions of which satisfy the equation (44), specifies by its action on O a field of (pseudo)orthonormal vielbeins and structure of (pseudo)Riemannian space. In particular, coefficients of affine connection in coordinate basis become equal to Christoffel symbols. The same structure of (pseudo)Riemannian space is specified on O by all the groups $\Gamma_T^{H'}$, the deformation functions of which are connected by transformations (46) from gauge group $SO(m, n-m)^g$.*

Arbitrary (pseudo)Riemannian structure on O can be specified in such fashion.

Through this theorem Klein's Erlangen Program is realized for geometric structure of (pseudo) Riemannian space.

This work makes a group-theoretic description of geometric structures of torsion-free affine connection and (pseudo)Riemannian space locally within the bounds of a single coordinate chart. This limitation can be lifted provided Lie pseudogroups are studied.

The fact that relations derived from the conditions of existence of certain groups have profound geometric meaning is another confirmation of fundamentality of the ideas of Klein's Erlangen Program in respect that geometry is defined completely by a group of congruencies.

References

[1] F. Klein, *Vergleichende Betrachtungen Uber Neuere Geometrische Forschungen (Erlanger Programm)*, Erlangen, 1872.

[2] E. Cartan *Group Theory and Geometry*, 1925.

[3] R. Sulanke, P. Wintgen, *Differentialgeometrie und Faserbundel*, Berlin, Veb Deutscher Verlag der Wissenschaften, 1972.

[4] L.V. Sabinin, *Methods of Nonassociativ Algebras in Differential Geometry,* 1981.

[5] S.E. Samokhvalov, *About the Setting of Connections in Fiber Bundles by the Acting of Infinite Lie Groups,* Ukrainian Math. J. **43** (1991) 1599-1603.

[6] S.E. SAMOKHVALOV, *Group-Theoretic Description of Gauge Fields*, Theor. Math. Phys. **76** (1988) 709-720.

[7] SAMOKHVALOV S.E., VANYASHIN V.S. *Group Theory Approach to Unification of Gravity with Internal Symmetry Gauge Interaction,* Class. Quantum Grav. **8** (1991) 2277 - 2282.

[8] S.E. SAMOKHVALOV, *Canonical Deformed Groups of Diffeomorphisms and Finite Parallel Transports in Riemannian Spaces,* arXive:0704.2980 [math.DG].